Proportional Relationships

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October 2014

PROPORTIONAL RELATIONSHIPS

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He will gladly respond to any feedback on the current piece sent to stanleyd@berkeley.edu, and also to suggestions for other content pieces.

Proportional Relationships

A. A simple example

Suppose a faucet is dripping steadily into a tub at such a rate that after 1 hour there are 6 liters of water in the tub. If the drip continues, the relationship can be expressed in a formula:

$w = 6t$ (w is accumulated water in liters, t is elapsed time in hours)

This is a simple example of a proportional relationship: The amount w of accumulated water ***is proportional to*** the time t it has been dripping. Specifically, the number of liters is 6 times the time in hours.

A proportional relationship is a relationship between two varying quantities in which one quantity is a constant multiple of another quantity.

Proportional relationships are the simplest, and also the most important kind of relationship in mathematics itself, and also in many naturally occurring situations. A list of common examples occurs in Section H below.

B. Describing proportional relationships

There are many different ways to describe a proportional relationship. We list them briefly here, using the faucet-dripping example from A.

1. A formula

A proportional relationship can be described by a formula. The formula in the faucet-dripping example is $w = 6t$. Here, t and w are variables, meaning simply that they can take on different values: they can vary. This formula tells us how to compute the amount w of water in liters from the time t elapsed in hours. (The CCSSM standards call this formula an equation. That is OK, because a formula is a type of equation.)

2. The constant of proportionality – a rate

The constant 6 in the formula $w = 6t$ is called the constant of proportionality of the proportional relationship. The number 6 is a rate: it tells us that the water is accumulating at the rate of "6 liters per hour".[1] A rate is always in the form "*n* A per B", where *n* is a number and each of A and B is a unit such as "liter" or "hour".

3. Constant multiple – constant ratio

The formula $w = 6t$ tells us that one quantity in a proportional relationship is a constant multiple of the other (*w* is 6 times t). A rearrangement of the formula is: $w/t = 6$. This tells us that the two quantities in a proportional relationship have a constant ratio (equal to 6). These are equivalent formulations.

Here are two equivalent mathematical definitions of a proportional relationship using this language:

- a relationship between two quantities is proportional if one quantity is a constant multiple of the other,

  or

- a relationship between two quantities is proportional if the two quantities have a constant ratio.

The constant ratio definition of a proportional relationship is the older one. It is why there is often a close association of the subjects of ratio and proportionality. See the domain in the standards: Ratio and Proportional Relationships. And recall the traditional topic in the middle school years: "ratio and proportion".

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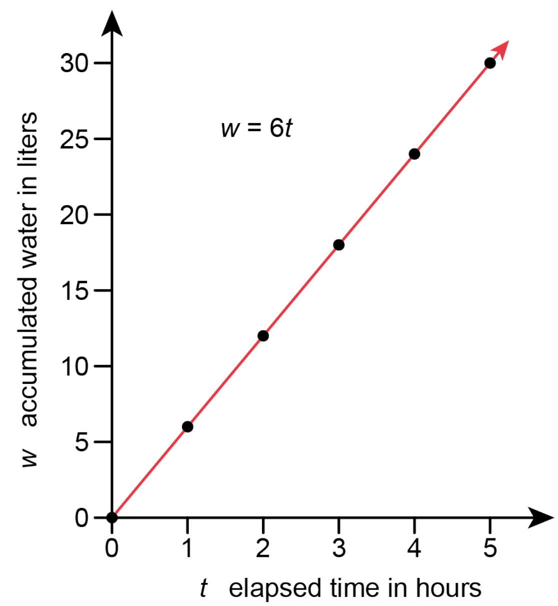
[1] The standards call this a *unit rate*. But, *unit rate* is redundant, because a rate as normally understood in math is already a *per unit quantity*. A description such as "18 liters in 3 hours" is not a rate. Rather it is information from which a rate could be derived. In this case, the rate is "6 liters per hour".

4. A graph (and its slope)

The graph of a proportional relationship such as $w = 6t$ is a straight line through the origin. The slope of the graph is the constant of proportionality (here, 6 liters per hour).



5. A table

Sample pairs of corresponding values of the two quantities in a proportional relationship are often shown in a table.

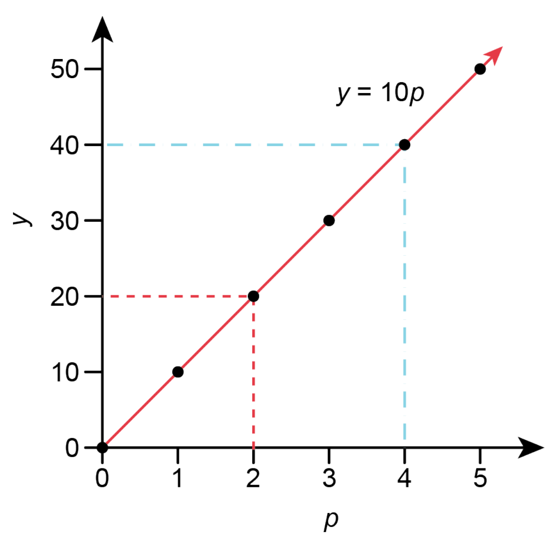
t	1	2	3	4	5	…	t
w	6	12	18	24	30	…	$6t$

When the value of t increases by 1, the corresponding value of w increases by 6. This is how the constant of proportionality shows up in a table.

C. The "doubling check" for proportional relationships

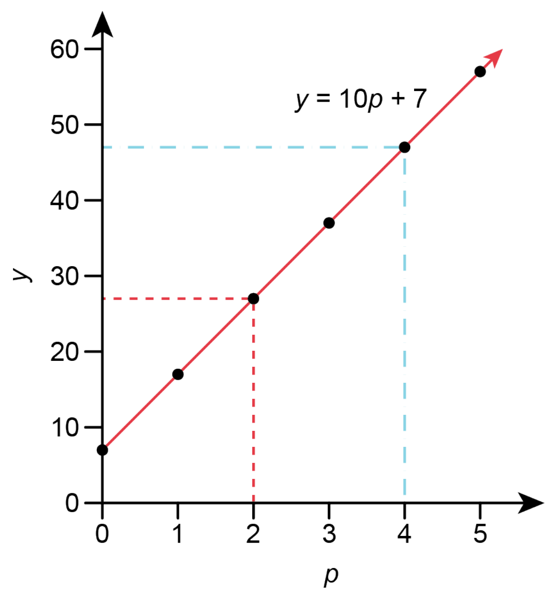
If twice as many hours have passed, twice as much water has accumulated. In short, if one quantity in a proportional relationship is doubled, the other is also doubled. This is the "doubling check" that distinguishes proportional relationships from all others: if you double the input, you double the output, but ***only if*** the relationship is proportional. We can show why this is true using either the formula or the graph:

- Start with the general formula: $y = kx$ for a proportional relationship. Pick an input (call it p). The output is $y = kp$. Now double the input from p to $2p$. The output is now $y = k(2p)$. Re-arrange this new output using commutativity of multiplication: $2(kp)$. Notice that that IS in fact double the first output kp.
- The fact that the doubling check works can also be seen directly on a graph. It depends on the fact that the graph is a straight line *through the origin* (0,0). (Use simple properties of similar triangles.)



We can also show that the doubling test fails when we have a relationship that is not proportional.

- Start with formula: $y = kx + b$, where $b \neq 0$. This is not proportional because of the b. Pick an input (call it p). The output is $y = kp + b$. Now double the input from p to $2p$. The output is now $y = k(2p) + b$. Re-arrange this new output using commutativity of multiplication: $2(kp) + b$. Now notice that this output is NOT $2(kp) + 2b$, which is double the first output $kp + b$.
- The fact that the doubling check fails for this non-proportional relationship can also be seen directly on a graph.



D. The role of "ratio and proportion"

The study of proportionality in school mathematics has been carried out for more than a century under the general heading "ratio and proportion". In many programs this subject has been reduced to a fixed procedure for solving a particular stereotyped kind of problem: setting up and solving a proportion. Unfortunately, in doing this, the idea of a proportional relationship itself is in danger of being lost.

Being able to set up and solve a proportion is important. We give a detailed illustration in the next section (E). But just as important is being able to connect this type of problem to the general concept of proportionality. We sketch this connection in the following section (F).

E. Proportions

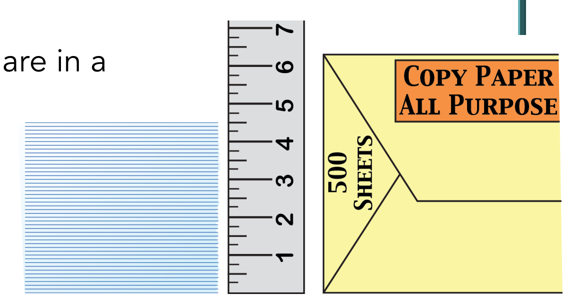
A "proportion" is an equation setting two ratios (or two fractions) equal. Typically, there is one unknown quantity (indicated with a variable such as *x* or *n* or some other symbol) and three known quantities. Setting up a proportion is useful in a situation if:

- we **know** there is a proportional relationship,
- we **don't** know the constant of proportionality,
- we know **both** numerical values for one pair of quantities in the relationship,
- we know **one** numerical value for another pair, and want to know the other.

Here is an example showing the process of setting up and solving a proportion:

Paper Stacks Problem

Suppose you want to know how many sheets are in a particular stack of paper, but don't want to count the pages directly. You have the following information:



• The given stack has height 4.50 cm.

• A ream of 500 sheets has height 6.25 cm.

Let *n* be the unknown number of sheets of paper in the given stack. We reason that the ratio of the number of sheets to the height of the stack must be the same for each stack.

So we set up the following equation (it is a proportion):

1. $\frac{n}{4.5} = \frac{500}{6.25}$ Each ratio gives the number of "sheets per centimeter" of height, which is the same for both stacks.

We want to find the value of *n*. We can do this by multiplying each side of the equation by 4.5: $n = \frac{500 \bullet 4.5}{6.25}$. Doing the arithmetic gives the answer, *n* = 360 sheets.

This is not the only way to set up the proportion. Here are three other ways, all valid:

2. $\frac{4.5}{n} = \frac{6.25}{500}$ Each ratio gives the number of "centimeters per sheet", which is the same for both stacks.

3. $\frac{n}{500} = \frac{4.5}{6.25}$ Each is a ratio between the size of the two stacks, small to large, the first in terms of sheets, the second in terms of centimeters. These ratios must be the same.

4. $\frac{500}{n} = \frac{6.25}{4.5}$ Each is a ratio between the size of the two stacks, large to small, the first in terms of sheets, the second in terms of centimeters. These ratios must be the same.

Solving each of these proportions leads to the same result: *n* = 360 sheets. This is the correct numerical result for this particular problem, but you would have to start all over if new numerical information were given for another stack of paper. Nowhere have we found a simple formula of the form $H = kn$, where *H* is the height and *n* is the number of sheets. Moreover, nowhere have we focused on the underlying proportional relationship here between the height of a stack and the number of sheets in a stack. This we do in the next section.

F. Proportion vs. proportional relationship

The focus in solving a proportion is the equality of two ratios that leads to a single numerical answer. The focus in analyzing a proportional relationship is in finding a constant of proportionality that represents an invariant of the situation, and leads us to a general solution to the problem.

Consider each of (1) to (4) from (E) above in turn:

1. If we compute the value of the common ratio 500/6.25 we get "80 sheets per cm". Thus we have the general formula $n = 80 \bullet h$. This says: The number of sheets in a stack is proportional to the height, with constant of proportionality 80 sheets per centimeter. This is a formula we could have used to solve the original problem given the data: 500 sheets take up 6.25 cm.

2. If we compute the value of the common ratio 6.25/500 we get "0.0125 cm per sheet". Thus we have the general formula $h = 0.0125 \bullet n$. This says: The height of a stack is proportional to the number of sheets, with constant of proportionality 0.0125 cm per sheet (the thickness of a single sheet). This is a formula we could have used for a different problem such as: If we make a stack of 50 sheets, how high will it be?

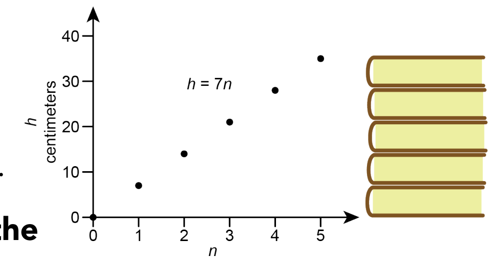
3. If we compute the value of the common ratio 4.5/6.25 we get 0.72. This is simply the ratio of the sizes of the two stacks: one is 72% as high as the other. But there is no proportional relationship for which this is a constant. We do not say: the small stack is proportional to the large stack.

4. If we compute the value of the common ratio 6.25/4.5 we get about 1.4. This is simply the ratio of the sizes of the two stacks: one is 1.4 times as high as the other. But there is no proportional relationship for which this is a constant. We do not say: the large stack is proportional to the small stack.

G. "PROPORTIONAL-OR-NOT" EXAMPLES:

Here are two situations: one leads to a proportional relationship, the other does not.

1. Stacks of books

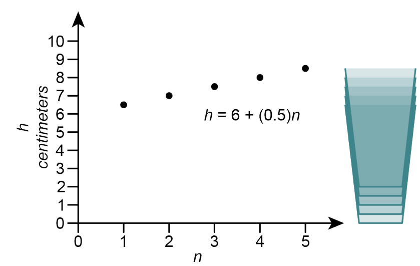


In a book warehouse suppose there are many stacks of textbooks (all the same kind).

"The height of a stack is proportional to the number of books in the stack."

This is true because the thickness of a book is the same (invariant) for all books. The thickness of one book is the constant of proportionality. If this thickness is 7 cm, the formula for the height h in terms of the number n of books is $h = (7)\bullet n$. Here, h is in centimeters, and "7 centimeters per book" is the constant of proportionality. A graph consists of separate dots along a line through the origin.

2. Stacks of cups



You have many paper cups (all the same kind). You create many stacks of these cups, using a different number of cups in each stack.

"The height of a stack is NOT proportional to the number of cups in the stack."

The height of a stack of 1 cup is the full height of a single cup. But the height of a stack of 2 cups only increases by the amount 1 cup sticks up, since it fits inside the first cup. So the increase in height is not uniform from 0 cups to 1 cup to 2 cups.

With measurements from real cups, we could make a table and draw a graph to show why this is NOT a proportional relationship. Alternatively, we could derive an explicit *formula* for the height of a stack of cups. Example, $h = 6 + (0.5)n$ is a formula for stacks of cups that are 6.5 cm high and that stick up 0.5 cm when stacked. The "6" destroys the form $y = kx$ for the formula of a true proportional relationship. So the height of a stack is NOT proportional to the number of cups in the stack.

H. Types of situations involving proportional relationships

1. Stacks

In this sort of situation there is a **stack** or **row** or **series** of many identical objects. Examples include a stack of paper. Here, $h = k\,n$, where h = height of stack and k = thickness of one piece.

2. Groupings

A grouping may be a table and its chairs, a team and its players, a spider and its legs, or in general, a "center" and its "satellites". Examples include a cafeteria with tables and chairs. If there are 10 chairs at a table, the relationship is $C = 10\,n$, where C is the number of chairs and n is the number of tables.

3. Unit conversions

Example: Since 1 inch = 2.54 centimeters, we have $C = 2.54\,I$, where the constant of proportionality has the unit "cm per inch". Here, C and I are the measurements, in cm and inches, respectively, of any given length.

4. Prices

There is a proportional relationship $p = Uq$ between the total price p paid for a product and the amount q purchased. The constant of proportionality U is the price of a "unit" amount and is called the unit price.

5. Charges

Often there is a fixed proportional relationship $c = Rm$ between an amount of money m and a one time "charge" c associated with that money. The constant of proportionality is almost always given as a percent. Examples include

a. a discount of 15% off a purchase price: $d = 0.15\,p$

b. a sales tax of 8.25% paid on a purchase price: $s = 0.0825\,p$

c. a bonus given as a percent of salary.

6. Slopes

This sort of situation involves a straight slope inclined to the horizontal. Over a given section of such a slope there is a proportional relationship $V = kH$ between the vertical "rise" V and the horizontal "run" H of the section. Examples of slopes include ramps , streets, and roofs.

7. Similar figures

A figure has "proportions", meaning the relationship of its parts. In a set of similar figures, the proportions are the same. For example, in a collection of 30-60-90 triangles, all the triangles are similar to one another; there is a proportional relationship h = 2s in between the hypotenuse *h* and the short side *s*. Other examples include:

a. The relationship $c = \Pi\, d$ between the circumference and the diameter of a circle.

b. The relationship $d = \sqrt{3}\ s$ between the side *s* of a cube and the diagonal *d*.

8. Enlargements/reductions

Between any pair of similar figures there is a size relationship between the length of a side in one figure and the length of the corresponding side in the other. For example, in a 155% enlargement of a picture, the enlarged picture is similar to the original picture, and there is a proportional relationship $e = 1.55\ s$ between the length *e* of any part of the enlargement and the length *s* of the corresponding part of the original. The proportionality constant 1.55 is the factor of enlargement.

9. Concentration

A "concentration" is a measure of a type of part-whole relationship. In a substance that has a given concentration of an active ingredient, there is a proportional relationship between the total amount of the substance and the amount of active ingredient. For example, if a substance has a concentration of active ingredient of 7.2%, the relationship $g = 0.072\ w$ between the amount *w* of the substance in a bottle and the amount *g* of active ingredient.

10. Sharing proportionally

In this sort of situation there is a certain amount of Q of some quantity that is to be divided proportionally up among a given number recipients according to some measure of the "size" of the recipient.

As an example, perhaps a certain amount m of state money is to be divided up proportionally among 6 schools according to the student population
n_1, n_2, n_3, n_4, n_5, n_6 of each school. In general the amount *m* varies from year to year. In terms of *m*, the amount of money received by school 1 is $m_1 = k_1$ m, where $k_1 = \dfrac{n_1}{n_1 + n_2 + n_3 + n_4 + n_5 + n_6}$, the constant of proportionality, is the proportion of the total student population that is in school 1. There is a corresponding constant of proportionality for each of the six schools.

11. Speeds

When an object is moving at a constant speed, there is a proportional relationship $d = R\,t$ between the distance *d* covered and the elapsed time *t*. Here, *R* is the speed.

Sometimes it is not an "object" that is moving, but a speed is involved nevertheless. For example, if a hose is filling a pool at a constant rate (in liters per minute), then the speed of the water in the hose is also constant, and can be figured out if the diameter of the hose is known.

12. Density

There are also "space rates" that tell how some quantity is distributed over space. Examples include:

a. population density (people per square mile)

b. coverage (grams per square centimeter of a coating)