Digging Deeper into Proportional Relationships

A check list of seven important ideas

This paper discusses seven important ideas related to proportional relationships. It is not intended to be a systematic and comprehensive treatment of the subject. Rather, it singles out seven important ideas that are often missed in treatments of proportionality. The treatments are not detailed or complete, but sketches to serve as a basis for discussion.

1. **Understanding when one quantity "is proportional to" another**
2. **Focusing on "constant ratio" rather than "equivalent ratios"**
3. **Clarifying the role of "per unit quantities"**
4. **Illustrating the requirement of "uniformity"**
5. **Differentiating two roles of "scaling" in proportional relationships**
6. **Providing a way to think about the "rate vs. ratio" issue.**
7. **Exploring many "situations" involving proportional relationships**

1. Understanding when one quantity "is proportional to" another

It is important to use the language *is proportional to* when talking about a proportional relationship. For example,

a. At a constant speed: distance traveled *is proportional to* time of travel.

b. In circles: the circumference *is proportional to* the radius.

c. The sales tax on an item *is proportional to* the given cost of the item.

This "is proportional to" language goes hand in hand with the formulas used to express the proportional relationship:

a'.	At a constant speed of 6 mph:	$d = 6t$
b'.	In circles:	$c = 2\pi r$
c'.	With sales tax at 8.5%:	$t = 0.085c$

Each of the numbers 6, 2π, and 0.85 is the *constant of proportionality* of the relationship.

A formal definition of the notion "is proportional to" can be based on the structure of these formulas:

> Definition 1: A variable quantity q *is proportional to* another variable quantity p if q is a multiple by a constant k of p:
>
> $q = kp$

Such quantities q and p are said to be in *a proportional relationship*.

The notion "is proportional to" is so important that it has its own symbol "$\propto$".

So in the above cases we can write:

a''.	At a constant speed:	$d = \propto t$	(d is proportional to t)
b''.	In circles:	$c = \propto r$	(c is proportional to r)
c''.	Sales tax:	$t = \propto c$	(t is proportional to c)

Use of the phrase "is proportional to" is universal in mathematics and its applications. Still, in many middle school treatments of proportional relationships, this phrase is seldom heard. Instead, the discussion is about "ratios" and "proportions".

2. Focusing on "constant ratio" rather than "equivalent ratios"

School mathematics treatments of ratio tend to focus on "equivalent" ratios. For example, the ratios 6:18 and 9:27 are equivalent. This is usually expressed in fraction notation:

$$\frac{6}{18} = \frac{9}{27}$$

In general, a statement that two ratios are equivalent is called a *proportion*:

$$\frac{p_1}{q_1} = \frac{p_2}{q_2}$$

However, once the curriculum moves on to proportional relationships, it is important to emphasize that a proportional relationship is a relationship between two ***variable*** quantities. In doing this, the notion of *two equivalent ratios* is awkward. The related notion of *a constant ratio* is much better suited. Specifically, we can say:

> In a proportional relationship, the two variable quantities
> have a ***constant ratio***.

Example (from part 1):

Motion at a constant speed of 6 mph	*d = 6t*	$\frac{d}{t} = 6$

On the left is the standard formula for expressing this proportional relationship. On the right is the same formula expressed as a ratio. In talking about this ratio, we say "d and *t* have a ***constant ratio***".

The constant ratio applies to an *infinite number* of pairs d and t. This suggests the following alternative definition of "proportional to":

> Definition 2: One variable quantity q is proportional to another variable quantity p if they have a ***constant ratio*** k:
>
> $$\frac{q}{p} = k$$

Notice that Definitions 1 and 2 are equivalent since the defining equations are algebraically equivalent.

As an aside, we note that Definition 2 can be useful in uniting all the different kinds of proportionality. We illustrate with three common types.

type	expressed in a formula	expressed as a constant ratio
direct proportionality	$y = kx$	$\frac{y}{x} = k$
inverse proportionality	$y = \frac{k}{x}$	$xy = k$
joint proportionality	$y = kxw$	$\frac{y}{xw} = k$

We see that each type of proportional relationship involves *a* ***constant ratio*** *of products of variable quantities.* Other important proportional relationships involve a combination of different types:

universal law of gravitation	$F = G\frac{m_1 m_2}{d^2}$	$\frac{Fd^2}{m_1 m_2} = G$

Note that in each case, the "constant" in the constant ratio is the constant of proportionality of the relationship.

3. Clarifying the role of "per unit quantities"

Consider the following examples of proportional relationships:

i. The distance sound travels *is proportional to* the time of travel.
   The speed of sound is about ***330 meters per second***.

ii. In gold nuggets, the mass *is proportional to* the volume.
   The density of gold is ***19.8 grams per cubic cm***.

iii. If apples sell for 3 pounds for $4, the cost *is proportional to* the weight.
   The unit price is ***3/4 pounds per dollar.***

In each case, the second statement gives the ***constant of proportionality*** of the proportional relationship. This constant is a "per unit quantity". That is, it tells us how much of one quantity there is for one unit of the other. (Note that a "unit" can be a "derived unit", as in the unit "cubic cm" in (iii).)

This use of "per" is widespread in comparing two quantities by dividing one quantity with a "unit" by another one with a "unit". It can also occur in a one-time comparison where there is no proportional relationship. For example, suppose we have 24 students and 6 computers. To see the relationship between these quantities we can divide 24 by 6 and get "***4 students per computer***". This is also a "per unit quantity".[1]

Understanding this feature of proportional relationships depends on a theory of "units analysis" appropriate for school, with its "algebra of units", including "canceling of units". It plays the same role as the more formal theory of "dimensional analysis" used in science. In short, "units analysis" and "per unit quantities" are an important part of understanding proportional relationships.

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[1] There are standard names (speed, density, unit price) for standard cases such as (i), (ii), and (iii). But it is not clear what name we would give to the number 4 here.

4. Illustrating the requirement of "uniformity"

Consider the following situation:

Situation: A faucet is dripping water steadily into a tub. After a time of 2 hours, there are 6 liters of water in the tub. It continues to drip steadily for several more hours.

The word "steadily" plays a key role here. It makes sense to interpret it as meaning that the volume of water in the tub is increasing "uniformly" over time. That is, during any time periods of the same length, the increase in the volume of water in the tub is the same.[2]

Under this interpretation, it would be expected that 12 liters of water would be expected after 4 hours, 3 liters after half an hour, etc..

In this situation we can say that "the accumulated volume is proportional to the elapsed time". Without the assumption of "uniformity", there is not a proportional relationship.

In every situation in which there is a proportional relationship, there is some sort of "uniformity" inherent in the situation, though this may take on rather different forms. For example, there is uniformity in examples (i) and (ii) in #3 above because of a physical law, and in (iii), it is "by decree".

Here are other examples:

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[2] It also makes sense to interpret the situation as involving both zero volume and zero time at the start of the drip. That is, we interpret time as "elapsed time" rather than clock time. Along with "uniformity", this "zero-zero" assumption for a situation is necessary for any proportional relationship based on the situation.

i. The volume of water in a rectangular tank *is proportional to* the depth. In fact $v = Ad$, where A is the cross sectional area of the tank. The uniformity is the result of a formula from solid geometry: volume = base area x height.

ii. The height of a stack of identical books *is proportional to* the number of books. The uniformity comes from the word "identical". If the stack was of books of differing thicknesses, there would not be a proportional relationship.

iii. In an enlargement on a photocopy machine set at 155%, lengths in the enlargement *are proportional to* the corresponding lengths in the original. Here the uniformity comes from the setting 155% of the machine.

5. Differentiating two roles of "scaling" in proportional relationships

A. True scaling behavior: scaling-in-tandem

Consider this example of a proportional relationship*:*

The height of a stack of identical books is proportional to the number of books. (See example (ii) in Section 4 above.)

In reasoning about this situation, it is clear that if we double the number n of books, the height h of the stack is doubled. Similarly, if we want to triple the height h of the stack, we need to triple the number n of books.

In general we can use a "scale factor" s to scale a quantity up or down. The example shows something important: In a proportional relationship between two quantities, the quantities "scale in tandem".

This behavior is clear if we use a formula for the relationship. If books are 3 cm thick, then a formula for height h in terms of count n is

(1) $h = 3n$

Multiplying both sides by s shows that scaling n by any factor s results in scaling the h by s also. This property is called "scaling-in-tandem".

Similarly, we can argue that if any two variable quantities y and x scale in tandem, they are related by a formula of the form $y = kx$, for some constant k. In fact, another way of characterizing a proportional relationship is to say that it is a relationship where the two related quantities "scale-in-tandem".

Note that in this example of scaling, the scale factor s is <u>not</u> a constant of proportionality. It is a way of stating a property of a proportional relationship. The property holds for *any* positive scale factor s.

B. Scaling when the constant of proportionality is a scale factor

The scaling behavior in (A) above is to be contrasted with cases where a scale factor is *the constant of proportionality of a proportional relationship*.

For example, given two similar figures, one 3 times the size of the other, corresponding lengths in the two figures *are proportional*: Specifically, lengths in one figure are 3 times the corresponding lengths in the other figure:

(2) $L' = 3L$,

Here, L is a length in one figure and L' is the corresponding "scaled" length in the larger figure.

Our main point is that the "scaling" by a factor of 3 in (2) is completely different from the "scaling in tandem" discussed in Part A above. Here the “scale factor" 3 is constant; it is the constant of proportionality of the proportional relationship.[3]

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[3] Although the constant of proportionality 3 in a case like (2) is usually thought of as being a dimensionless scale factor, it also makes sense to think of it as a "per unit quantity": "3 cm in the larger figure per cm in the smaller figure".

6. Providing a way to think about the "rate vs. ratio" issue.

A question often arises: What is the difference between a rate and a ratio?

There are several common answers. For example, "In a ratio the units of the two quantities are the same, in a rate they are different." But none of these answers gets deeply enough into the issue. Here we suggest a "structural" way of thinking about this issue.

In developing this "structural" picture, it helps to include ratios with more than two terms, since there is no controversy in referring to these as "ratios". For example, consider the three-term ratio

1 : 2 : 8

With this ratio we can associate these 9 quotients:

1 ÷ 1 = 1	1 ÷ 2 = 0.5	1 ÷ 8 = 0.125
2 ÷ 1 = 2	2 ÷ 2 = 1	2 ÷ 8 = 0.25
8 ÷ 1 = 8	8 ÷ 2 = 4	8 ÷ 8 = 1

If this ratio (1:2:8) was a recipe:

salt (tsp) : flour (oz.) : water (cups),

then these quotients have meanings like this:

0.5 tsp salt per oz. of flour

0.25 oz. flour per cup of water

These are "per unit quantities". They act like things we call "rates", such as a constant speed of 6 miles per hour.

So we can say this:

(1) A "ratio-like quantity" is an n-tuple, where n = 2, 3, 4, ...

(2) A "rate-like quantity" is any quotient in a ratio-like quantity.

Whatever we call them, the quantities in (1) and (2) are clearly important, and also clearly ***very different***. (We will shorten these to "ratio" and "rate" below.)

Most ratios in the school curriculum are 2-tuples. A ratio 5:2 (a 2-tuple) has two rates associated with it (quotients): 5÷2 = 2.5 and 2÷5 = 0.4.

So if, say, there is a student: computer ratio 5:2, then the two rates are:

- 2.5 students per computer
- 4 tenths of a computer per student

In short: An *n*-term ratio has n^2 quotients associated with it, and these quotients act like rates.

7. Exploring many "situations" involving proportional relationships

The best way to talk clearly about a proportional relationship is to be explicit about the "situation" it is based on. For any proportional relationship, two things are required of the situation:

a. There are two quantities in the situation that "co-vary".

The idea is that when one quantity changes, the other changes too. More specifically, the two quantities change in such a way that their ratio is constant.

b. There is something about the situation that is "uniform".

(See part 4 above.) The kind of uniformity is different for different situations. It may be the constant rate of speed of some process, or the constant thickness of a book in stacks of identical books. Whatever the uniformity is, it is the basis for the constant of proportionality.

Exploring many real situations involving proportional relationships can contribute to a general understanding of such relationships. To give the idea of the variety of such situations, here is an initial listing of seven different types:

1. Stacks

In this sort of situation there are stacks of many identical objects. An example stacks of paper: the height of the stack is proportional to the number of sheets.

2. Unit conversions

Example: Since 1 inch = 2.54 centimeters, we have C = 2.54 I, saying that a measurement C in cm is proportional to the same measurement I in inches.

3. Prices

There is a proportional relationship $p = Uq$ between the total price p paid for a product and the amount q purchased. The constant of proportionality U is the price of a “unit” amount and is called the "unit price".

4. Sales tax

At a sales tax rate of 8.25% paid on a purchase price, there is a proportional relationship $t = 0.0825\ p$ between the tax t paid and the quoted price p.

5. Slopes

In ramps with slope s there is a proportional relationship $V = sH$ between the vertical "rise" V and the horizontal "run" H of the ramp.

6. Shape relationships in an infinite set of similar figures

There is a proportional relationship $c = \pi\ d$ between the circumference c and the diameter d of a circle.

7. Size relationships in a pair of similar figures

Between any pair of similar figures there is a size relationship between the length of a side in one figure and the length of the corresponding side in the other. For example, in a 155% enlargement of a picture, there is a proportional relationship $e = 1.55\ o$ between the length e of any part of the enlargement and the length o of the corresponding part of the original.